Remark 2.122. If $\mathscr{C}$ is left or right closed monoidal, we may simplify the Day convolution product using the Yoneda lemma for coends:

$$\begin{aligned} F * G &= \int^{a,b} \mathscr{C}(a \otimes b, -) \otimes_{\Bbbk} Fa \otimes_{\Bbbk} Gb \cong \int^{a,b} \mathscr{C}(a, [b,-]_{\ell}) \otimes_{\Bbbk} Fa \otimes_{\Bbbk} Gb \\ &\overset{(2.8.1)}{\cong} \int^{b} F([b,-]_{\ell}) \otimes_{\Bbbk} Gb, \end{aligned} \tag{2.8.3}$$

$$\begin{aligned} F * G &= \int^{a,b} \mathscr{C}(a \otimes b, -) \otimes_{\Bbbk} Fa \otimes_{\Bbbk} Gb \cong \int^{a,b} \mathscr{C}(b, [a,-]_{r}) \otimes_{\Bbbk} Fa \otimes_{\Bbbk} Gb \\ &\overset{(2.8.1)}{\cong} \int^{a} Fa \otimes G([a,-]_{r}). \end{aligned} \tag{2.8.4}$$

Example 2.123. Consider a $\Bbbk$-linear monoidal category $\mathscr{X}$ with a single object x. Then $A := \mathrm{End}_{\mathscr{X}}(x)$ is a commutative algebra and $[\mathscr{X}, \mathsf{Vect}] \cong \text{Mod-}A$. Let $F, G \colon \mathscr{X} \longrightarrow \mathsf{Vect}$ be two functors. Writing $M := Fx$ and $N := Gx$ for the corresponding modules over A, and $an := na$ for all $a \in A$ and $n \in N$, we obtain the following using Equation (2.8.3) and the definition of coends:

$$(F * G)x \cong {}^{M \otimes_{\Bbbk} N}\!\big/\!{}_{\langle ma \otimes_{\Bbbk} n - m \otimes_{\Bbbk} an \mid m \in M, n \in N, a \in A \rangle} = M \otimes_A N.$$

Thus, one recovers the tensor product of modules over commutative algebras.

Theorem 2.124 ([Day71])**.** *For any monoidal category $\mathscr{C}$, the category $[\mathscr{C}^{\mathrm{op}}, \mathsf{Vect}]$ is closed monoidal with Day convolution as its tensor product, $\mathscr{C}(1,-)$ as its unit, and the internal homs given for all $F, G \colon \mathscr{C}^{\mathrm{op}} \longrightarrow \mathsf{Vect}$ by*

$$[F,G]_{\ell} := \int_{a,b} \mathsf{Vect}(\mathscr{C}(- \otimes a, b), \mathsf{Vect}(Fa, Gb)), \tag{2.8.5}$$

$$[F,G]_{r} := \int_{a,b} \mathsf{Vect}(\mathscr{C}(a \otimes -, b), \mathsf{Vect}(Fa, Gb)). \tag{2.8.6}$$

Remark 2.125. If the monoidal category $\mathscr{C}$ is closed, then the formulas for the internal homs of $[\mathscr{C}, \mathsf{Vect}]$ may be simplified by means of the Yoneda lemma:

$$\begin{aligned} [F,G]_r &= \int_{a,b} \mathsf{Vect}(\mathscr{C}(a \otimes -, b), \mathsf{Vect}(Fa, Gb)) \\ &\cong \int_{a,b} \mathsf{Vect}(\mathscr{C}(a \otimes -, b) \otimes_{\Bbbk} Fa, Gb) \cong \int_b \mathsf{Vect}\Big(\int^a \mathscr{C}(a \otimes -, b) \otimes_{\Bbbk} Fa, Gb\Big) \\ &\cong \int_b \mathsf{Vect}\Big(\int^a \mathscr{C}(a, [-,b]_{\ell}) \otimes_{\Bbbk} Fa, Gb\Big) \overset{(2.8.1)}{\cong} \int_b \mathsf{Vect}(F[-,b]_{\ell}, Gb), \end{aligned} \tag{2.8.7}$$

$$[F,G]_\ell \cong \int_b \mathsf{Vect}\Big(\int_a \mathscr{C}(a,[-,b]_r)\otimes_{\Bbbk} Fa, Gb\Big) \overset{(2.8.1)}{\cong} \int_a \mathsf{Vect}(F[-,b]_r, Gb). \tag{2.8.8}$$

Remark 2.126. Whenever we treat $\widehat{\mathscr{C}^{\mathrm{op}}} := [\mathscr{C}, \mathsf{Vect}]$ as a (closed) monoidal category, we implicitly equip with the convolution tensor product. Analogously, one may define a closed monoidal structure on $\widehat{\mathscr{C}} := [\mathscr{C}^{\mathrm{op}}, \mathsf{Vect}]$. Note, however, that in this case we cannot simplify the internal hom in the same way as in Theorem 2.125; this would require the category $\mathscr{C}^{\mathrm{op}}$, as opposed to $\mathscr{C}$ itself, to be closed monoidal.

The convolution structure is particularly well-behaved on representables:

$$\mathscr{C}(-,x) * \mathscr{C}(-,y) = \int^{a,b\in\mathscr{C}} \mathscr{C}(-,a\otimes b)\otimes_{\Bbbk} \mathscr{C}(a,x)\otimes_{\Bbbk}\mathscr{C}(b,y) = \mathscr{C}(-,x\otimes y)$$

This connection extends to the entire functor category, see [Day71; IK86].

Proposition 2.127. *For a monoidal category $\mathscr{C}$, the Yoneda embedding*

$$よ\colon \mathscr{C} \longrightarrow \widehat{\mathscr{C}} = [\mathscr{C}^{\mathrm{op}}, \mathsf{Vect}], \qquad x \longmapsto \mathscr{C}(-,x) \tag{2.8.9}$$

is a strong monoidal functor.

2.9 (Co)completions

In this subsection we give a brief—informal—account of the results regarding the monoidal pseudofunctoriality of cocompletions and the resulting (co)completion operations for monoidal and module categories. We refer to [Kel05; KS06] for generalities on (co)limits and (co)completions. We implicitly assume all categories and functors to be $\Bbbk$-linear.

Let Φ be a class of diagrams. We say that a category is Φ-*cocomplete* if it admits colimits of functors with domain in Φ, and we say that a functor is Φ-*cocontinuous* if it preserves such colimits.

Definition 2.128. A monoidal category $\mathscr{C}$ is called *separately Φ-cocontinuous* if $\mathscr{C}$ is Φ-cocomplete and its tensor product is separately Φ-cocontinuous.

Similarly, for a Φ-cocomplete monoidal category $\mathscr{D}$, a $\mathscr{D}$-module category $\mathscr{M}$ is said to be separately Φ-cocontinuous if $\mathscr{M}$ is Φ-cocomplete and the action $-\triangleright_{\mathscr{M}} =$ is separately Φ-cocontinuous.