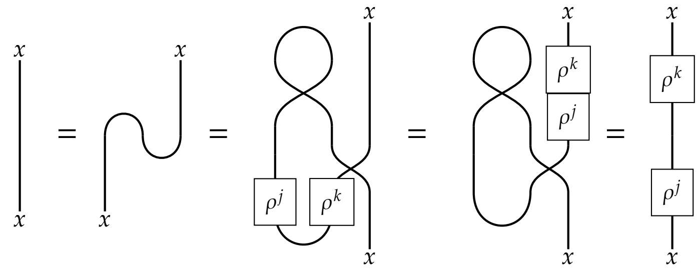
study the coevaluation of x, fix a half-braiding $\chi_{x^2,-}\colon x^2 \otimes - \longrightarrow - \otimes x^2$ on x^2. Due to Theorem 4.46, it is determined by

$$\chi_{x^2,x} = \sigma_{x^2,x} \circ \big((\sigma^i_{x,x} \circ (\rho^j_x \otimes \rho^k_x)) \otimes \rho^l_x\big), \qquad \text{where } i,j,k,l \in \mathbb{Z}_2.$$

Now suppose $\mathrm{coev}_x\colon 1 \longrightarrow x^2$ lifts to a morphism in $\mathsf{Z}(\mathscr{C})$, where x^2 is equipped with this half-braiding. Equation (4.2.9) and the self-duality of $\sigma_{x,x}$ imply that $\sigma_{x,x} \circ \mathrm{coev}_x = \mathrm{coev}_x$ and $\mathrm{ev}_x \circ \sigma_{x,x} = \mathrm{ev}_x$; we compute



Therefore, $j = k$ and $\zeta_{(x^2,\chi_{x^2,-})} = \mathrm{id}^2_x$ or $\zeta_{(x^2,\chi_{x^2,-})} = \rho^2_x$, implying naturality. A similar argument for the evaluation of x concludes the proof. □

By Theorem 4.44, the Picard heap of $\mathsf{A}(\mathscr{C})$ can have at most two elements. However, the above theorem constructs a third pivotal structure on $\mathsf{Z}(\mathscr{C})$; this implies our desired result.

Theorem 4.50. *The pivotal structure ζ of $\mathsf{Z}(\mathscr{C})$ is not induced by the Picard heap of $\mathsf{A}(\mathscr{C})$. In particular, the map $\iota\colon \operatorname{Pic}\mathsf{A}(\mathscr{C})/\operatorname{Pic}\mathsf{SZ}(\mathscr{C}) \longrightarrow \operatorname{Piv}\mathsf{Z}(\mathscr{C})$ of Theorem 4.33 is not surjective.*

[D]ie Monade, das
unvollkommenste aller Wesen.

Arthur Schopenhauer;
Die Kunst, Recht zu behalten

MONADIC TANNAKA–KREIN RECONSTRUCTION

Bimonads and Hopf monads are a vast generalisation of bialgebras and Hopf algebras. They naturally arise in the study of (rigid) monoidal categories and topological quantum field theories, see amongst others [KL01; Moe02; BV07; BLV11; TV17]. Recall that the definition of a bialgebra object necessarily requires a braided monoidal category as a base, in order to write down what it means for the multiplication to be a morphism of comonoids. However, in general the category of endofunctors is not braided, so the naïve notion of bialgebras does not generalise to the monadic setting and needs to be adjusted. One possible way of overcoming this problem was introduced and studied by Moerdijk under the name *Hopf monad*, [Moe02].[13] Here, the structure morphisms of an oplax monoidal functor serve as a substitute of the comultiplication and counit. There are other, sometimes non-equivalent, notions of Hopf monad, see [Boa95; MW11]. We follow [Moe02], with a slight terminology change due to [BV07; BLV11]. This definition aims to generalise the paradigmatic example of the free–forgetful adjunction of a Hopf algebra. Thus, equipping a monad with the prefixes "bi-" or "Hopf" refers to additional structure or properties put on its Eilenberg–Moore category.

[13] As remarked in [Moe02], this concept is strictly dual to that of monoidal comonads, which are studied in [Boa95].

More generally, a monadic interpretation of module categories was given by Aguiar and Chase under the name *comodule monad*, [AC12]. A comodule monad over a bimonad generalises the notion of a comodule algebra over a bialgebra, see Theorem 5.14 below. The main result of this chapter extends the reconstruction results of [AC12, Proposition 4.1.2] and [TV17, Lemma 7.10]:

Theorem 5.28. *Let $\mathscr{C}$ and $\mathscr{D}$ be monoidal categories, and suppose that $\mathscr{M}$ and $\mathscr{N}$ are right $\mathscr{C}$- and $\mathscr{D}$-module categories, respectively. Let $F\colon \mathscr{C} \rightleftarrows \mathscr{D} :U$ be an oplax monoidal adjunction. Lifts of an adjunction $G\colon \mathscr{M} \rightleftarrows \mathscr{N} :V$ to a comodule adjunction are in bijection with lifts of $V\colon \mathscr{N} \longrightarrow \mathscr{M}$ to a strong comodule functor.*

From the proof of this result one immediately obtains an analogue of Kelly's doctrinal adjunction result, [Kel74], for $\mathscr{C}$-module categories.