Our notation for the Eilenberg–Moore category already ascribes a certain "terminal" quality to it; see Theorem 2.23.

Remark 2.13. For any monad T, the T-algebras and their morphisms form a category: the *Eilenberg–Moore category of* T. We shall denote it by $\mathscr{C}^T$.

The Eilenberg–Moore category of T is also often called the *category of T-algebras* or, following for example [BV07], the *category of modules* over T. We use all three terminologies interchangeably.

A monad is intimately connected to its Eilenberg–Moore category.

Example 2.14. There is a 2-category $\mathbb{M}\mathsf{on}(\mathbb{C}\mathsf{at})$ of monads in $\mathbb{C}\mathsf{at}$, [Str72, § 1]. The inclusion 2-functor maps a category to its identity monad:

$$\mathbb{C}\mathsf{at} \longrightarrow \mathbb{M}\mathsf{on}(\mathbb{C}\mathsf{at}), \qquad \mathscr{C} \longmapsto (\mathrm{Id}_{\mathscr{C}}, \mathrm{id}_{\mathrm{Id}_{\mathscr{C}}}, \mathrm{id}_{\mathrm{Id}_{\mathscr{C}}}).$$

By assumption, $\mathbb{C}\mathsf{at}$ *admits the construction of algebras*: there exists a right adjoint to the above functor:

$$\mathbb{M}\mathsf{on}(\mathbb{C}\mathsf{at}) \longrightarrow \mathbb{C}\mathsf{at}, \qquad (T\colon \mathscr{C} \longrightarrow \mathscr{C}, \mu, \eta) \longmapsto \mathscr{C}^T,$$

where $\mathscr{C}^T \in \mathbb{C}\mathsf{at}$ is the Eilenberg–Moore category of T. Using the previous 2-adjunction, one proves that to every monad (T, μ, η) on $\mathscr{C}$ there exist an *Eilenberg–Moore adjunction* $F^T\colon \mathscr{C} \longrightarrow \mathscr{C}^T$ and $U^T\colon \mathscr{C}^T \longrightarrow \mathscr{C}$, such that

$$T = U^T F^T, \qquad \mu = F^T \varepsilon U^T, \qquad \eta = \eta,$$

where $\eta\colon 1_{\mathscr{C}} \Longrightarrow U^T F^T$ and $\varepsilon\colon F^T U^T \Longrightarrow 1_{\mathscr{C}^T}$ are the unit and counit of the Eilenberg–Moore adjunction. We shall call F^T and U^T the *free* and *forgetful* functor *associated to* T, respectively.

For the following definition, we follow [BV07; TV17].

Definition 2.15. Suppose that T and S are two monads on the category $\mathscr{C}$. A *morphism of monads* between T and S is a natural transformation $f\colon T \Longrightarrow S$, such that the following diagrams commute

$$\begin{tikzcd} \mathrm{Id} \arrow[r, "\eta^T"] \arrow[dr, "\eta^S"'] & T \arrow[d, "f"] \\ & S \end{tikzcd} \qquad \begin{tikzcd} TT \arrow[r, "Tf"] \arrow[d, "\mu^T"'] & TS \arrow[r, "fS"] & SS \arrow[d, "\mu^S"] \\ T \arrow[rr, "f"'] & & S \end{tikzcd} \tag{2.2.1}$$

Remark 2.16. The terminology of Theorem 2.15 is slightly non-standard. What we call a morphism of monads is often called a *oplax* (or colax) monad morphism, see for example [Str72, § 1].

Remark 2.17. One can define a monad in any bicategory $\mathbb{B}$ by considering (C, t, η, μ), where $C \in \mathbb{B}$ is an object, $t\colon C \longrightarrow C$ is a 1-cell, and $\eta\colon \mathrm{Id}_C \Longrightarrow t$ and $\mu\colon tt \Longrightarrow t$ are 2-cells, satisfying relations analogous to Theorem 2.9. An oplax morphism of monads from (C, t, η^t, μ^t) to (D, s, η^s, μ^s) then consists of a 1-cell $u\colon C \longrightarrow D$ and a 2-cell $\phi\colon ut \Longrightarrow su$, subject to identities reminiscent of Diagram (2.2.1). A *lax morphism of monads* involves a 1-cell $u\colon C \longrightarrow D$ and a 2-cell $\phi\colon su \Longrightarrow ut$, satisfying similar properties.

The following example sheds some additional light on this terminology.

Example 2.18. Monads can alternatively be defined as lax functors—in the sense of Theorem 2.5—from the terminal 2-category $\heartsuit$ to $\mathbb{C}\mathsf{at}$. Unravelling this definition, a lax functor $\mathfrak{T}\colon \heartsuit \longrightarrow \mathbb{C}\mathsf{at}$ consists of:

- an object assignment $\mathfrak{T}\colon \operatorname{Ob}\heartsuit \longrightarrow \operatorname{Ob}\mathbb{C}\mathsf{at}$, sending the unique object $*$ to a category $\mathscr{C}$;
- a functor $\mathfrak{T}(*,*)\colon \heartsuit(*,*) \longrightarrow \mathbb{C}\mathsf{at}(\mathscr{C},\mathscr{C})$ from the terminal 1-category $\heartsuit(*,*)$ to the category of endofunctors on $\mathscr{C}$, sending the unique 1-morphism $\mathrm{id}_*\colon * \longrightarrow *$ to $T\colon \mathscr{C} \longrightarrow \mathscr{C}$ and the unique 2-morphism $1_{\mathrm{id}_*}\colon \mathrm{id}_* \Longrightarrow \mathrm{id}_*$ to the identity natural transformation $T \Longrightarrow T$;
- a 2-cell $\mathfrak{T}_2\colon \mathfrak{T}\mathrm{id}_* \otimes \mathfrak{T}\mathrm{id}_* \Longrightarrow \mathfrak{T}\mathrm{id}_*$, which we write as $\mu\colon TT \Longrightarrow T$; and
- a 2-cell $\mathfrak{T}_0\colon 1_{\mathfrak{T}(*)} \Longrightarrow \mathfrak{T}\mathrm{id}_*$ that we write as $\eta\colon \mathrm{Id}_{\mathscr{C}} \Longrightarrow T$.

Recall that $\otimes$ is the horizontal composition in $\mathbb{B}$.

The properties of Theorem 2.5 for $\mathfrak{T}_2$ and $\mathfrak{T}_0$ translate to the associativity and unitality properties of μ and η. In this setting, a morphism of monads becomes an oplax transformation in the sense of Theorem 2.6.

Example 2.19. A monad T on $\mathscr{C}$ has another canonical category associated to it: its *Kleisli category* $\mathscr{C}_T$. On objects, it is given by $\operatorname{Ob}(\mathscr{C}_T) := \operatorname{Ob}(\mathscr{C})$, and for $x, y \in \mathscr{C}_T$ we have $\mathscr{C}_T(x,y) := \mathscr{C}(x, Ty)$. Composition is defined by

$$\begin{aligned} \circ\colon \mathscr{C}_T(y,z) \times \mathscr{C}_T(x,y) &\longrightarrow \mathscr{C}_T(x,z) \\ (g,f) &\longmapsto \big(x \xrightarrow{f} Ty \xrightarrow{Tg} T^2z \xrightarrow{\mu_z} Tz\big). \end{aligned}$$

Proposition 2.20. *Let T be a monad on a category $\mathscr{C}$. There exists an adjunction*

$$\begin{tikzcd} \mathscr{C} \arrow[r, shift left=2, "F_T"] \arrow[r, phantom, "\perp"] & \mathscr{C}_T \arrow[l, shift left=2, "U_T"] \end{tikzcd}$$

where F_T is identity on objects and sends $f \in \mathscr{C}(x,y)$ to $\eta_y \circ f$, and U_T sends x to Tx and $f \in \mathscr{C}(x,y)$ to $\mu_y \circ Tf$.