The Drinfeld centre

Classically, the centre construction is used to build a braided monoidal category from a monoidal one, see for example [EGNO15, Chapter 7]. Throughout especially Chapters 4 and 5, we work in a slightly more general setting, see for example [GNN09; BV12; HKS19; FH23; Kow24].

Analogously to Theorem 2.53, one can show that $\sigma_{M,1} = \mathrm{id}_M$ is implied by the hexagon axiom.

Definition 2.75. Let $\mathscr{C}$ be a monoidal category, $\mathscr{M}$ a $\mathscr{C}$-bimodule category, and $m \in \mathscr{M}$. A *half-braiding* on m is a natural isomorphism $\sigma_{-,=} \colon - \triangleleft = \Longrightarrow - \triangleright =$, such that for all $x, y \in \mathscr{C}$ we have $\sigma_{M,1} = \mathrm{id}_M$ and

$$\sigma_{m,x\otimes y} = (\mathrm{id}_x \triangleright \sigma_{m,y}) \circ (\sigma_{m,x} \triangleleft \mathrm{id}_y).$$

Definition 2.76. The *centre* of a $\mathscr{C}$-bimodule category $\mathscr{M}$ is the category $\mathsf{Z}(\mathscr{M})$ defined as follows:

- Objects are pairs $(m, \sigma_{m,-})$ of an object $m \in \mathscr{M}$ and a half-braiding $\sigma_{m,-}$.
- A morphism $f \colon (m, \sigma_{m,-}) \longrightarrow (n, \sigma_{n,-})$ consists of an $f \in \mathscr{M}(m, n)$ that commutes with the half-braidings:

$$(\mathrm{id}_x \triangleright f) \circ \sigma_{m,x} = \sigma_{n,x} \circ (f \triangleleft \mathrm{id}_x), \qquad \text{for all } x \in \mathscr{C}.$$

There is a canonical forgetful functor $U^{(M)} \colon \mathsf{Z}(\mathscr{M}) \longrightarrow \mathscr{M}$. Unlike classical representation theory where the centre of a bimodule is a subset of the bimodule, $U^{(M)}$ need not be injective on objects in general.

Example 2.77. The centre $\mathsf{Z}(\mathscr{C})$ of the regular bimodule of a monoidal category $\mathscr{C}$, see Theorem 2.44, is the *Drinfeld centre* of $\mathscr{C}$. The tensor product is defined by $(x, \sigma_{x,-}) \otimes (y, \sigma_{y,-}) := (x \otimes y, \sigma_{x\otimes y,-})$, with

$$\sigma_{x\otimes y,z} := (\sigma_{x,z} \otimes \mathrm{id}_y) \circ (\mathrm{id}_x \otimes \sigma_{y,z}), \qquad \text{for all } z \in \mathscr{C}.$$

The centre is braided monoidal, with braiding given by gluing together the respective half-braidings. The hexagon axioms follow from the definition of the half-braiding and the tensor product of $\mathsf{Z}(\mathscr{C})$.

Example 2.78. Let $H \in \mathsf{Vect}$ be a finite-dimensional Hopf algebra. Then the category of left-left Yetter–Drinfeld modules of Theorems 2.33 and 2.54 is, as a braided monoidal category, equivalent to the Drinfeld centre of the category ${}_H\mathsf{Vect}$ of left H-modules, [Dri87; Yet90].

Further, by [Dri87] there exists another Hopf algebra $D(H)$, the *Drinfeld double* of H, that has $\mathsf{Z}({}_H\mathsf{Vect}) \simeq {}^H_H\mathscr{YD}$ as its category of modules. We refer to [Kas98, Definition IX.4.1] for a complete definition, and to [Kas98, Theorem XIII.5.1] for a proof of the above equivalence.

Remark 2.79. The category of anti-Yetter–Drinfeld modules of Theorem 2.42 is also the category of modules over a certain Hopf algebra: the *anti-Drinfeld double* [CMZ97; Sch99]. In analogy to how the anti-Yetter–Drinfeld modules are a module category over the Yetter–Drinfeld modules, the anti-Drinfeld double is a comodule algebra over the Drinfeld double.

For the next result, recall Theorem 2.71 for iterated duals.

Proposition 2.80 ([JS91, Lemma 7]). *The centre of a (strict) rigid category $\mathscr{C}$ is (strict) rigid: for all $(x, \sigma_{x,-}) \in \mathsf{Z}(\mathscr{C})$, we have that $U^{(\mathsf{Z})}\big({}^{\vee}(x, \sigma_{x,-})\big) = {}^{\vee}x$ and $U^{(\mathsf{Z})}\big((x, \sigma_{x,-})^{\vee}\big) = x^{\vee}$. Moreover, for every $n \in \mathbb{Z}$ and $x \in \mathsf{Z}(\mathscr{C})$, the equality $\sigma_{(x)^n,(y)^n} = (\sigma_{x,y})^n$ holds for all $y \in \mathsf{Z}(\mathscr{C})$.*

2.5 Linear and abelian categories

We shall pay special attention to $\Bbbk$-linear categories, for some field $\Bbbk$. Indeed, $\Bbbk$-linear functor categories encompass many phenomena in representation theory; see for example Section 3.2 and Chapter 9. We shall denote the category $\Bbbk$-vector spaces by Vect, and write vect for the full subcategory of finite-dimensional $\Bbbk$-vector spaces.[3]

[3] This will be a general theme throughout the thesis: whenever there exists a full subcategory of "finite-dimensional objects" inside of a lager category of all objects, the finite-dimensional subcategory will start with a lower case letter, and we use a capital for the larger category.

Definition 2.81. A $\Bbbk$*-linear category* is a category enriched in Vect

Example 2.82. Let $\mathscr{C}$ be an ordinary category. Its *linearisation* $\Bbbk\mathscr{C}$ has the same objects, and the space of morphisms between two objects $a, b \in \Bbbk\mathscr{C}$ is $\Bbbk\mathscr{C}(a, b) := \operatorname{span}_{\Bbbk}\mathscr{C}(a, b)$. To define the composition in $\Bbbk\mathscr{C}$, note that for any $a, b, c \in \Bbbk\mathscr{C}$ there is a unique linear map

$$- \circ = \colon \Bbbk\mathscr{C}(b, c) \otimes_{\Bbbk} \Bbbk\mathscr{C}(a, b) \longrightarrow \Bbbk\mathscr{C}(a, c),$$

such that $g \circ f = gf$ for all $g \in \mathscr{C}(b, c)$ and $f \in \mathscr{C}(a, b)$.

Let $\iota\colon \mathscr{C} \longrightarrow \Bbbk\mathscr{C}$ be the functor that is the identity on objects and maps any morphism to the corresponding basis vector. For any functor $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ whose codomain is a $\Bbbk$-linear category, we obtain a commuting triangle:

$$\begin{tikzcd} \mathscr{C} \arrow[rr, "F"] \arrow[dr, "\iota"'] & & \mathscr{D} \\ & \Bbbk\mathscr{C} \arrow[ur, "\exists!\, G \text{ (linear)}"'] & \end{tikzcd}$$

As a consequence, if we endow the ordinary functor category $[\mathscr{C}, \mathsf{Vect}]$ with the pointwise $\Bbbk$-linear structure, we obtain an isomorphism of linear categories between it and the category of $\Bbbk$-linear functors from $\Bbbk\mathscr{C}$ to Vect.